JOURNAL OF ALGEBRA 177, 755–767 (1995)

Double Coset Density in Reductive Algebraic Groups

Jonathan Brundan

Department of Mathematics, Imperial College, London, SW7 2AZ, United Kingdom

Communicated by Jan Saxl

Received September 24, 1994

INTRODUCTION

Throughout, G will denote a linear algebraic group over an algebraically closed field K of characteristic $p \geq 0$. Writing $R_u(G)$ for the largest normal connected unipotent subgroup of G, we take a *reductive* group G to be a (not necessarily connected) algebraic group with $R_u(G^0) = 1$. Our main result is as follows.

THEOREM 1. *Let G be a connected reductive group and $X < G$ a proper reductive subgroup. Then, there is no dense (X, X)-double coset in G, and in particular there are infinitely many (X, X)-double cosets.*

By way of contrast, if P is a parabolic subgroup then there are at most $|W|$ double cosets of P in G, where W is the Weyl group of G. Also, if w_0 is the longest word of W, then the double coset Pw_0P is dense in G.

When G acts transitively on a variety Ω, define the permutation rank of G on Ω to be the number of G_α-orbits in Ω, for some point $\alpha \in \Omega$. We now give an immediate consequence of Theorem 1.

COROLLARY. *Let G be a connected reductive group acting transitively on a non-trivial affine variety Ω, such that the orbit map is separable. Then there is no dense G_α-orbit in Ω, for $\alpha \in \Omega$, and in particular the permutation rank of G on Ω is infinite.*

This can be deduced from Theorem 1 as follows. The orbit map induces a closure-preserving correspondence between (G_α, G_α)-double cosets in G and G_α-orbits in Ω; so, we just need to show that G_α is reductive. Separability implies $G/G_\alpha \cong \Omega$ by [Hum, 12.4], so G/G_α is also affine. Now, [Ri, Theorem A] implies that G_α is indeed reductive.

Theorem 1 arose from attempts to prove an analogue of a well known theorem of Seitz [Se, Theorem 2] regarding finite Chevalley groups. This says that if X is a subgroup of a finite (possibly twisted) Chevalley group $G = G(q)$, with a uniform bound on the number of (X, X)-double cosets

0021-8693/95 $12.00

in G for large q, then $U^P \leq X \trianglelefteq P$ for some parabolic subgroup $P \leq G$, with U a maximal unipotent subgroup of P. Our result implies in particular that if X is a subgroup of a connected reductive group G with finitely many double cosets, then X lies in a parabolic subgroup of G. The proof given in [Se] involves character theory to show that if there is a bound on the number of (X, X)-double cosets, then the number of (X, U)-double cosets is also bounded. He then goes on to show [Se, Theorem 1] that the result stated follows from this second bound.

We prove the analogue in an entirely different way, focusing on certain invariant polynomials that arise quite naturally. Write $K[G]$ for the coordinate ring attached to the affine variety G. For $X, Y \leq G$, let

$$^{X}K[G]^{Y} = \{f \in K[G] \mid f(xgy) = f(g), \forall x \in X, y \in Y, g \in G\}.$$

We make some preliminary remarks concerning double cosets. Let X, Y be closed subgroups of G. Then, a double coset XgY is just an orbit for the obvious action of $X \times Y$ on G. In particular, the closure $\overline{XgY}$ is a union of double cosets, and a double coset of minimal dimension in $\overline{XgY}$ will be closed. There is a natural closure-preserving bijection between (X, Y)-double cosets in G and X-orbits in G/Y (similarly, Y-orbits in G/X); sometimes it is more convenient to work with these.

The following lemma explains the relevance of the polynomials $^{X}K[G]^{Y}$ to this situation.

LEMMA 0.1. *If $X, Y \leq G$ are closed subgroups, with G connected, and $^{X}K[G]^{Y} \neq K$, then*

(i) *there is no dense (X, Y)-double coset in G;*

(ii) *there are infinitely many closed (X, Y)-double cosets in G.*

Proof. Take a non-constant polynomial $f \in {}^{X}K[G]^{Y}$ and any $g \in G$. Then f is constant on all (X, Y)-double cosets, so XgY certainly lies in the closed subset of G defined by the vanishing of the polynomial $f - f(g)$. This is not all of G as f is not constant on all of G, so XgY is not dense in G, giving (i).

Now for (ii), just note that f must take infinitely many values as G is connected. For each value c of f, the closed set C of G defined by the vanishing of $f - c$ is a union of double cosets. Then, a double coset of minimal dimension in C will be closed by the above remarks. ∎

Our main result will follow easily from Theorem 2 and an application of this lemma.

THEOREM 2. *Let G be a simple algebraic group and $X < G$ a proper connected reductive subgroup. Suppose that at least one of the following holds:*

(i) *$X' = X$ and X lies in some proper parabolic subgroup of G;*

(ii) *$X' \neq X$;*

(iii) $T \leq X$ *and* $\Sigma(X)$ *is a closed subsystem of* $\Sigma(G)$;

(iv) $\dim X \geq \frac{1}{2} \dim G$.

Then, ${}^{X}K[G]^{X} \neq K$.

In Section 2, we prove Theorem 2 when (i), (ii), or (iii) holds by giving direct constructions for non-constant polynomials in ${}^{X}K[G]^{X}$. To prove the result in the case (iv), we use the dimension bound to list the possibilities for X, applying Seitz's classification of maximal connected subgroups of the simple algebraic groups. Of course, this dimension bound will be sufficient to deduce Theorem 1, where we are only considering density of double cosets. This deduction is given in Section 3.

We conjecture that ${}^{X}K[G]^{X} \neq K$ for any proper connected reductive subgroup X of G when G is simple.

1. PRELIMINARIES

We first establish some notation and give some general lemmas. Fix a Borel subgroup B of G and a maximal torus $T < B$, and let $U = R_u(B)$. Let $W = N_G(T)/T$ be the Weyl group of G and $X(T) = \hom(T, K^*)$ be the character group of T. When G is reductive, the choice of T and B determines a root system $\Sigma(G)$ and a base $\Pi(G)$ for $\Sigma(G)$. For $\alpha \in \Sigma(G)$, let U_α be the corresponding root subgroup of G. Let $\Pi(G) = \{\alpha_1, \dots, \alpha_r\}$ and $\{\lambda_1, \dots, \lambda_r\}$ be the corresponding fundamental weights. If G is simple and λ is a dominant weight, denote the rational irreducible KG-module of highest weight λ by $V_G(\lambda)$. Generally, when V is a rational KG-module and $\lambda \in X(T)$, let V_λ be the corresponding weight space of V.

For G classical, we use the notation $G = \mathrm{Cl}(V)$ to indicate that G is a connected classical algebraic group with natural module V. If $(G, p) = (B_n, 2)$, take V to be the associated $2n$-dimensional symplectic module. When $G \neq SL(V)$, let N_i denote the connected stabilizer in G of a non-degenerate subspace of V of dimension i with $i \leq \frac{1}{2} \dim V$, and when $(G, p) = (SO(V), 2)$, let N_1 denote the connected stabilizer of a nonsingular 1-space.

The next lemma gives some straightforward properties of ${}^{X}K[G]^{X}$.

LEMMA 1.1. *Let X be a closed subgroup of G with ${}^{X}K[G]^{X} \neq K$.*

(i) *Suppose θ: $G' \to G$ is a surjective homomorphism (of algebraic groups) sending $X' < G'$ onto X. Then, ${}^{X'}K[G']^{X'} \neq K$.*

(ii) *Suppose $G = XY$ is a factorisation of G, with Y a closed subgroup. Then, ${}^{X \cap Y}K[Y]^{X \cap Y} \neq K$.*

(iii) *Suppose Y is conjugate to X in G. Then, ${}^{Y}K[G]^{Y} \neq K$.*

Proof. Take $f \in {}^{X}K[G]^{X}$, not constant on G. For (i), $\theta^* f \in {}^{X'}K[G']^{X'}$ will not be constant on G' as θ is surjective. For (ii), f is not constant on Y

as $G = XY$. So, the restriction of f to Y is certainly in ${}^{X\cap Y}K[Y]^{X\cap Y}$ and will not be constant on Y. Part (iii) is clear, as conjugating X gives rise to translation of functions in ${}^{X}K[G]^{X}$. ∎

The Mumford conjecture, proved in [Hab], gives a non-constructive means of proving the existence of invariant polynomials in ${}^{X}K[G]^{X}$. Many of the proofs below are valid without needing the strength of the Mumford conjecture, but we use it freely in what follows since it simplifies calculations. In fact, we use the following simple consequence of the Mumford conjecture:

LEMMA 1.2 [Ri, Lemma 1.4]. *Let X be a reductive group acting morphically on an affine variety V, with C, D two disjoint closed X-stable subsets of V. Then there exists $f \in K[V]^{X}$ such that $f(x) = 0$ for $x \in C$ and $f(x) = 1$ for $x \in D$.*

We also use the fact that G/X is an affine variety if X is a reductive subgroup of G [Ri, Theorem A], which is another consequence of the Mumford conjecture. We now give some simple consequences of this lemma.

COROLLARY 1.3. *Let X be a reductive subgroup of G.*

(i) *${}^{X}K[G]^{X} \neq K$ if and only if G has at least two disjoint closed (X, X)-double cosets.*

(ii) *If X lies in some proper closed subgroup P of G and ${}^{X}K[P]^{X} \neq K$, then ${}^{X}K[G]^{X} \neq K$.*

(iii) *If $N_G(X)$ is strictly larger than X, then ${}^{X}K[G]^{X} \neq K$.*

(iv) *Let $\theta : G \to G'$ be a homomorphism of algebraic groups sending $X < G$ onto $X' < G'$, with $\ker\theta \leq X$. If ${}^{X}K[G]^{X} \neq K$ then ${}^{X'}K[G']^{X'} \neq K$.*

Proof. (i) ($\Rightarrow$) This follows as in Lemma 0.1 (ii).

($\Leftarrow$) Let C, D be two disjoint closed (X, X)-double cosets in G. Then their images under the quotient morphism $q : G \to G/X$ are also closed in the variety $V = G/X$, which is affine as X is reductive. Now, applying the lemma, there is a non-constant $f \in {}^{X}K[G/X]$. It just remains to pull back to $q^{*}f$ which will be in ${}^{X}K[G]^{X}$.

(ii) By Lemma 0.1, there are two closed (X, X)-double cosets in P, so there certainly are in G. Now apply (i).

(iii) Take $z \in N_G(X) - X$. Then, the double cosets X and $XzX = zX$ are disjoint and closed, so apply (i).

(iv) Homomorphisms of algebraic groups are open maps, so any closed subset of G which is a union of $\ker\theta$-cosets will be mapped to a closed subset of G'. This holds in particular for (X, X)-double cosets as $\ker\theta \leq X$. Now, take two disjoint closed (X, X)-double cosets; these are mapped to two disjoint closed (X', X')-double cosets. X' is also reductive, and the result follows by (i). ∎

Remarks. (I) We give an example to show (i)–(iii) need not hold if X is not reductive. Let $G = SL(V)$, P be the stabiliser of a line $\langle v \rangle$, with $0 \neq v \in V$, and X be the stabiliser of the point v. It is easy to see that Xw_0X contains Bw_0B, and so is a dense double coset, where w_0 is the longest element of W. Hence, ${}^XK[G]^X = K$. However, by the argument in Proposition 2.1, ${}^XK[P]^X \neq K$, so there are even infinitely many closed (X, X)-double cosets in G.

(II) Note that (iv) can be proved without assuming that X is reductive providing θ is *separable*. Then, $\theta(G)$ is isomorphic to $G/\ker \theta$. Any non-constant $f \in {}^XK[G]^X$ is constant on $\ker \theta$-cosets as $\ker \theta \leq X$, so induces a non-constant polynomial in ${}^{X'}K[G']^{X'}$, by the universal property of quotients.

The final result in this section gives all possibilities (G, X) with G simple, X a maximal connected reductive subgroup, and $\dim X \geq \frac{1}{2} \dim G$, to be used in proving Theorem 2 (iv). There are only a few possibilities, given by [LSS]. In that paper, Seitz's classification of maximal connected reductive subgroups of the simple algebraic groups, together with some information on modules of small dimension for the simple groups, is applied to list the possibilities. When $p = 3$, write $\tilde{A}_2$ for the subgroup of G_2 generated by the short root groups.

PROPOSITION 1.4. *Let G be a simple algebraic group and $X < G$ a maximal connected reductive subgroup with* $\dim X \geq \frac{1}{2} \dim G$. *In the case G classical, let V be the natural module for G. Then (G, X) are as follows*:

G	X	Conditions
$Sp(V)$, $SO(V)$	N_k	
$SL(V)$	$Sp(V)$	
$Sp(V)$	$SO(V)$	$p = 2$
SO_8	B_3	$V \downarrow X = V_X(\lambda_3)$
$SO_7(p \neq 2)$, $Sp_6(p = 2)$	G_2	$V \downarrow X = V_X(\lambda_1)$
G_2	A_2	
G_2	$\tilde{A}_2$	$p = 3$
F_4	B_4	
F_4	C_4	$p = 2$
E_6	F_4	
E_7	A_1D_6	
E_8	A_1E_7	

Proof. For G classical this is [LSS, Lemma 5.1], while for G exceptional it follows from [LSS, Proposition 2.3]. ∎

2. PROOF OF THEOREM 2

PROPOSITION 2.1. *Let X be a proper reductive subgroup of an algebraic group G. If $X = X' < P$ for some proper parabolic subgroup P of G, then ${}^XK[G]^X \neq K$. In particular, Theorem 2(i) holds.*

Proof. By a theorem of Chevalley, we can find a rational representation V for G and a line $L = \langle v \rangle$ such that $P = \mathrm{stab}_G(L)$. Now, P cannot fix v, for otherwise there is a bijective morphism from the complete variety G/P to Gv. By [Hum, 6.1], Gv is then complete and closed in V, so as complete affine varieties have dimension 0 and Gv is connected, $Gv = \{v\}$, a contradiction. Thus, there is some $h \in P$ with $hv \neq v$.

Now, the map $f: P \to K^*$ defined by $gv = f(g)v$ is a homomorphism of algebraic groups. Since $X < P$ and $X' = X$, $X < \ker f$. Hence, X fixes v. Now, if $x_1, x_2 \in X$ and $g \in P$, then $f(x_1 g x_2) = f(x_1)f(g)f(x_2) = f(g)$, so $f \in {}^XK[P]^X$. Also, $f(h) \neq 1 = f(1)$, so f is not constant on all of P. Now, the result follows by Corollary 1.3(ii). ∎

The next lemma provides the key to Theorem 2(ii).

LEMMA 2.2. *Let δ be an automorphism of an algebraic group G and suppose X is a reductive subgroup of G^δ, where $G^\delta = \{x \in G \mid \delta(x) = x\}$. If there is some $s \in G$ with $1 \neq s\delta(s)^{-1}$ semisimple, then ${}^XK[G]^X \neq K$.*

Proof. Conjugacy classes of semisimple elements are always closed, so the conjugacy class in G of $s\delta(s)^{-1}$, $\mathrm{Cl}_G(s\delta(s)^{-1})$, is closed. Now, $C = \overline{\mathrm{Cl}_X(s\delta(s)^{-1})}$ lies in $\mathrm{Cl}_G(s\delta(s)^{-1})$, so C is certainly disjoint from $D = \{1\}$. As C is a union of X-conjugacy classes both C and D are X-stable. So, by Lemma 1.2 applied to the conjugation action of X on G, there is some $f \in K[G]$ constant on X-conjugacy classes in G, taking value 0 on C and 1 on D.

Now consider the polynomial $f'(g) = f(g\delta(g)^{-1})$. By construction, $f'(1) = 1$ and $f'(s) = 0$, so f' is not constant on G. It remains to show $f' \in {}^XK[G]^X$. Let $g \in G$, $x \in X$. Then, $f'(gx) = f(gx\delta(x)^{-1}\delta(g)^{-1}) = f(g\delta(g)^{-1}) = f'(g)$ since $\delta(x) = x$. Also, $f'(xg) = f(xg\delta(g)^{-1}x^{-1}) = f(g\delta(g)^{-1}) = f'(g)$ since f is constant on X-conjugacy classes in G. ∎

Remark. There is a direct construction here avoiding the Mumford conjecture. The polynomial $f(g) = \mathrm{tr}_V(g\delta(g)^{-1})$, where V is any rational KG-module, is easily seen to be left and right X-invariant. This needs to be non-constant, which is generally the case, for some choice of V. There are exceptions though, usually in small characteristic. For example, if $(G, X, p) = (SL, Sp, 2)$, then f is constant on G for all choices of V.

PROPOSITION 2.3. *Let X be a connected reductive subgroup of a connected reductive group G, with $Z(X)$ not central in G. Then ${}^XK[G]^X \neq K$. In particular, Theorem 2(ii) holds.*

Proof. By conjugating X if necessary, we may assume $Z(X) \leq T$. Pick $z \in Z(X) - Z(G)$. As $C_G(N_G(T)) = Z(G)$, we can find some $s \in N_G(T)$ that does not centralise z, so $szs^{-1}z^{-1} \neq 1$. As s normalises T and $z \in T$, this element lies in T, so is semisimple. Now, apply Lemma 2.2 with $\delta(g) = zgz^{-1}$. Then, $X \leq G^{\delta}$ and $1 \neq s\delta(s)^{-1}$ is semisimple as required.

Finally, to see that Theorem 2(ii) follows, note that if X is reductive with $X' \neq X$, then X is not semisimple by [Hum, 27.5], so $Z(X)$ has positive dimension while $Z(G)$ is finite. ∎

We now describe a construction for polynomials in ${}^XK[G]^X$, to be applied to prove Theorem 2(iii). We use some non-standard notation. Suppose $\rho : G \to GL(V)$ is a rational representation, where $V = W \oplus W'$ (vector space direct sum). Then, for any $g \in G$, define $B_W(g) \in \mathrm{End}(W)$ to be the composition of the restriction of $\rho(g)$ to W followed by projection $V \to W$ along the direct sum. $B_W(g)$ is just the block matrix corresponding to W when $\rho(g)$ is written with respect to a basis corresponding to the direct sum.

LEMMA 2.4. *Let $X < G$ be a closed subgroup of a simple algebraic group G, and let ρ: $G \to GL(V)$ be a rational representation which splits as a direct sum of KX-modules, $V = W \oplus W'$. For $g \in G$, define $B_W(g) \in \mathrm{End}(W)$ and $B_{W'}(g) \in \mathrm{End}(W')$ as above. Then, $f(g) = \det(B_W(g))\det(B_{W'}(g))$ lies in ${}^XK[G]^X$.*

Proof. Let $x, y \in X$ and $g \in G$. First we show $f(xgy) = f(x)f(g)f(y)$. For, take any $w_0 \in W$, and let $y.w_0 = w_1, g.w_1 = w_2 + w_2', x.w_2 = w_3, x.w_2' = w_3'$, where $w_1, w_2, w_3 \in W$, $w_2', w_3' \in W'$. Then, $B_W(x)B_W(g)B_W(y).w_0 = w_3$, whilst $xgy.w_0 = w_3 + w_3'$. Hence, $B_W(xgy).w_0 = w_3 = B_W(x)B_W(g)B_W(y).w_0$. Similarly, $B_{W'}(xgy) = B_{W'}(x)B_{W'}(g)B_{W'}(y)$, so $f(xgy) = f(x)f(g)f(y)$ by the multiplicative property of determinants.

Now G is simple so certainly $\det(\rho(g)) = 1$, for all $g \in G$. For $x \in X$, $\rho(x)$ is just a block diagonal matrix with $B_W(x)$ and $B_{W'}(x)$ as the blocks, with respect to a basis corresponding to the direct sum. So, $\det(\rho(x)) = \det(B_W(x)) \det(B_{W'}(x)) = f(x) = 1$ and $f(xgy) = f(x)f(g)f(y) = f(g)$, as required. ∎

Recall that a closed subsystem $\Phi \subset \Sigma$ is a root system such that whenever a $\mathbb{Z}$-linear combination of roots in Φ is a root in Σ, then it is also a root in Φ.

PROPOSITION 2.5. *Let X be a proper connected reductive subgroup of the simple group G. If $T \leq X$ and the root system $\Sigma(X)$ of X is a closed subsystem of $\Sigma(G)$, then ${}^XK[G]^X \neq K$. So, Theorem* 2(iii) *holds.*

Proof. We have $X = \langle T, U_\alpha \mid \alpha \in \Sigma(X)\rangle$. For λ, $\mu \in X(T)$, define $\mathrm{Shape}(\lambda) = \mathrm{Shape}(\mu)$ if $\lambda - \mu$ is a $\mathbb{Z}$-linear combination of roots in $\Sigma(X)$. This gives an equivalence relation on $X(T)$. Call the equivalence classes shapes, following the language of [ABS]. Then, the roots in $\Sigma(G)$ of shape [0] are precisely the roots of X, since $\Sigma(X)$ is a closed subsystem of $\Sigma(G)$. So, there is more than one shape in $\Sigma(G) \cup \{0\}$.

Now, let V be the adjoint module of G and S be any shape in $\Sigma(G) \cup \{0\}$. Define V_S to be the span of weight spaces ΣV_α, where the sum is over all $\alpha \in \Sigma(G) \cup \{0\}$ of shape S. We claim V_S is a KX-submodule of V. It is certainly T-stable, so we need to check $U_\beta V_\alpha \subset V_S$ for $\beta \in \Sigma(X)$ and $\mathrm{Shape}(\alpha) = S$. But, $U_\beta V_\alpha \subset \Sigma_{k \in \mathbb{Z}^+} V_{\alpha + k\beta}$, and $\mathrm{Shape}(\alpha + k\beta) = \mathrm{Shape}(\alpha)$ as required.

Hence, $V = \Sigma V_S$, as S runs over all shapes in $\Sigma(G) \cup \{0\}$, is actually a direct sum of KX-modules. As in Lemma 2.4, define $f(g) = \Pi \det(B_{V_S}(g))$ as S runs over all shapes. Then f is in ${}^XK[G]^X$. It remains to show f is not constant on G. For this, we show there are shapes $S \neq S'$ containing roots of G of the same length. This will suffice, for W acts transitively on roots of the same length, so there exist $\alpha \in S$ and $n \in N_G(T)$ with $nV_\alpha \subset V_{S'}$. But then $\det(B_{V_S}(n)) = 0$ so $f(n) = 0 \neq f(1)$.

If there are at least three shapes or if $\Sigma(G)$ has only one root length, the claim is certainly true. So, we may assume there are precisely two shapes, $S = \Sigma(X) \cup \{0\}$ and S', and roots in S have different length from roots in S'. As $\Sigma(X)$ is a closed subsystem, S must consist of long roots, whilst S' consists of short roots. Now for all cases except $(\Sigma(G), \Sigma(X)) = (B_n, D_n)$ (taking $n \geq 3$ in the case $\Sigma(G) = C_n$) there are short roots α, β with $\alpha + \beta$ also a short root. But then $\mathrm{Shape}(\alpha + \beta) = \mathrm{Shape}(\alpha)$ implies β is a $\mathbb{Z}$-combination of roots in $\Sigma(X)$, so lies in $\Sigma(X)$, a contradiction.

This leaves the case of $(\Sigma(G), \Sigma(X)) = (B_n, D_n)$. This is just the embedding N_1 in $SO(V)$ when $p \neq 2$, dealt with below in Case 1 in Proposition 2.6. When $p = 2$, apply a surjective morphism $B_n \to C_n$ which maps (G, X) to $(Sp(V), SO(V))$. In Case 4 of Proposition 2.6, we construct non-constant invariant polynomials for the latter embedding, which can then be lifted to G using Lemma 1.1(i). ∎

Remark. Subsystem subgroups generally have centres, when this result also follows from Proposition 2.3. But there are some exceptions in small characteristics, requiring the alternative construction given in this proof.

Finally, we complete the proof of Theorem 2, proving the final case.

PROPOSITION 2.6. *Let X be a proper connected reductive subgroup of the simple group G with* dim $X \geq \frac{1}{2}$ dim G. *Then* ${}^{X}K[G]^{X} \neq K$. *So, Theorem* 2(iv) *holds.*

Proof. If X lies in a proper parabolic subgroup of G, the result follows by Theorem 2(i) and (ii). So, we may assume X is a maximal connected reductive subgroup of G, when the possibilities are given by Proposition 1.4. We now eliminate each case in turn. Note that by Lemma 1.1(i) it is sufficient to verify each case when G is of adjoint type, applying the surjective homomorphism $G \to \mathrm{Ad}\ G$ to deduce other types. However, when G is classical, we take G as $\mathrm{Cl}(V)$. Then, providing the polynomials constructed actually lie in ${}^{XZ}K[G]^{XZ}$ where $Z = Z(G) = \ker \mathrm{Ad}$, Lemma 1.3(iv) will imply the result for G adjoint.

Case 1: $(G, X) = (Sp(V), N_k), (SO(V), N_k)$. Let X be the stabiliser of the subspace $W < V$. With the exception of $(G, X, p) = (SO(V), N_1, 2)$, W is non-degenerate so $V = W \oplus W^{\perp}$ gives a splitting of V as a direct sum of KX-modules. W and $W^{\perp}$ are even Z-stable, so applying Lemma 2.4, $f(g) = \det(B_W(g)) \det(B_{W^{\perp}}(g))$ lies in ${}^{XZ}K[G]^{XZ}$. To show f is not constant on all of G, just note G acts transitively on vectors of equal length, so there are certainly $g \in G$ and $w \in W$ with $g.w \in W^{\perp}$. Then, $f(g) = 0 \neq f(1)$ so f is not constant on G as required.

For the exceptional case, $W = \langle w \rangle$ is a non-singular line. As $p = 2$, X even fixes w. Define $f(g) = (w, gw)$ where $(\ , \ )$ is the symplectic form preserved by G. We claim $f \in {}^{X}K[G]^{X}$; f is certainly not constant on G, so this will complete the case. But, for $x_1, x_2 \in X$, $x_i w = w$ so $f(x_1 g x_2) = (w, x_1 g x_2 w) = (x_1 w, x_1 g w) = (w, gw) = f(g)$, and f is left and right X-invariant.

Case 2: $(G, X) = (SO_8, B_3), (SO_7, G_2)(p \neq 2), (Sp_6, G_2)(p = 2)$. If $G = SO_8$, $X = N_1$, Case 1 implies that ${}^{X}K[G]^{X} \neq K$. Apply a triality automorphism of G sending $B_3 \to N_1$, together with Lemma 1.1(i) to deduce the result for $(G, X) = (SO_8, B_3)$.

Now, let $G = SO_8$, $X = B_3$, $Y = N_1$. Then, $Y \cong SO_7$, $X \cap Y = G_2$, and $XY = G$ by [LSS, Theorem B]. By the previous paragraph, ${}^{X}K[G]^{X} \neq K$. Now apply Lemma 1.1(ii) to conclude the result for $(Y, X \cap Y) = (SO_7, G_2)$. Finally, use the existence of a surjective morphism (in characteristic 2) sending $Sp_6 \to SO_7$ and Lemma 1.1(i) to deduce the result for $(G, X, p) = (Sp_6, G_2, 2)$.

Case 3: $(G, X) = (SL(V), Sp(V)), (E_6, F_4)$. In these cases, X is the centralizer of a graph automorphism δ of G, which may be chosen to stabilise T. So taking any $s \in T - X$, $s\delta(s)^{-1}$ will be semisimple. The result now follows immediately from Lemma 2.2.

Remark. In [Law], Lawther gives a parametrisation of (X, X)-double cosets in G here, exploiting certain "folding" symmetry in the Dynkin diagrams.

Case 4: $(G, X, p) = (Sp(V), SO(V), 2)$. Our treatment of this case is adapted from [I], which actually gives a precise parametrisation of (X, X)-double cosets. Let G be the stabiliser of a non-singular symplectic form $(\, , \,)$ in $GL(V)$, with $e_1, \dots, e_n, f_1, \dots, f_n$ a symplectic basis for V relative to $(\, , \,)$. Then, G acts on Ω, the set of all quadratic forms on V compatible with $(\, , \,)$ by $R^g(v) = R(gv)$, for $R \in \Omega$, $v \in V$. Let $Q \in \Omega$ be the unique quadratic form with $Q(e_i) = Q(f_i) = 0$ for all i, and X be the stabiliser of Q in G.

LEMMA 2.7. *Given $R \in \Omega$, there exists a unique vector $\mu(R) \in V$ such that $Q(v) + R(v) = (v, \mu(R))^2$, for all $v \in V$. Moreover, the function $\mu : \Omega \to V$ has the property that $\mu(R^x) = x^{-1}\mu(R)$, for $x \in X$.*

Proof. Let $v = \sum a_i e_i + \sum b_i f_i$. Then, noting $p = 2$, we expand both sides of the equation:

$$Q(v) + R(v) = \sum_{i=1}^{n} a_i^2 R(e_i) + \sum_{i=1}^{n} b_i^2 R(f_i);$$

$$(v, \mu(R))^2 = \sum_{i=1}^{n} a_i^2 (e_i, \mu(R))^2 + \sum_{i=1}^{n} b_i^2 (f_i, \mu(R))^2.$$

We want these to be equal for all v, forcing $(e_i, \mu(R))^2 = R(e_i)$ and $(f_i, \mu(R))^2 = R(f_i)$. So the required vector must be

$$\mu(R) = \sum_{i=1}^{n} R(e_i)^{1/2} f_i + \sum_{i=1}^{n} R(f_i)^{1/2} e_i. \tag{1}$$

Now, for all $v \in V$, $x \in X$,

$$\begin{aligned}(v, \mu(R^x))^2 &= Q(v) + R^x(v)\\ &= Q(xv) + R(xv) \quad (\text{as } Q^x = Q)\\ &= (xv, \mu(R))^2\\ &= (v, x^{-1}\mu(R))^2.\end{aligned}$$

Hence, $\mu(R^x) = x^{-1}\mu(R)$. ∎

Define $f \in K[G]$ by $f(g) = Q(\mu(Q^g))^2$. Then, f is certainly left X-invariant as $Q^x = Q$ for $x \in X$. But it is also right X-invariant since for

$x \in X$, $g \in G$,

$$f(gx) = Q(\mu(Q^{gx}))^2 = Q(x^{-1}\mu(Q^g))^2 = Q(\mu(Q^g))^2 = f(g),$$

applying Lemma 2.7. To see f is indeed a polynomial, expand $Q(\mu(Q^g))^2$ using (1) to obtain

$$f(g) = \sum_{i=1}^{n} Q^g(e_i)Q^g(f_i). \tag{2}$$

It remains to show f is not constant on all of G. But G acts transitively on Ω, and $R \in \Omega$ can be defined by picking arbitrary values $R(e_i)$, $R(f_i)$ on a basis and extending to V. This makes it clear from (2) that f can take any value in K.

Case 5: $(G, X) = (G_2, A_2), (F_4, B_4), (E_7, A_1D_6), (E_8, A_1E_7)$; $(G, X, p) = (F_4, C_4, 2), (G_2, A_2, 3)$.

The first four cases follow from Theorem 2(iii), since in each case, the roots of X form a closed subsystem of $\Sigma(G)$. The remaining cases follow from (F_4, B_4) and (G_2, A_2) respectively. In either case, there exists a graph automorphism of G in the exceptional characteristic (only a bijective morphism of algebraic groups) sending $C_4 \to B_4$ or $\tilde{A}_2 \to A_2$. Now apply Lemma 1.1(i). ∎

This completes the proof of Theorem 2.

3. PROOF OF THEOREM 1

It remains to deduce Theorem 1 from Theorem 2. The remaining case to be dealt with is that of "diagonal" subgroups of semisimple groups:

LEMMA 3.1. *Let $G = G_1G_2$ be a connected semisimple group, with $Z(G) = 1$, splitting as a direct product of two simple factors G_i. Suppose $X < G$ is a closed subgroup that projects onto both G_1 and G_2. Then, there is no dense (X, X)-double coset in G.*

Proof. Let $\pi_1 : G \to G_1$ and $\pi_2 : G \to G_2$ be the projections. Note $XG_2 = G$ as π_1 is onto, so $X \cap G_1 \lhd G_1$. Since $Z(G_1) = 1$, G_1 is simple as an abstract group, so $X \cap G_1 = 1$ or G_1. But the latter cannot happen since then $G = XG_1 = X$. Hence $X \cap G_1 = 1$. Then, $\theta_1 = \pi_1 \downarrow X$ is an isomorphism (of abstract groups) $X \to G_1$, and similarly for $\theta_2 = \pi_2 \downarrow X$. In particular, $G_1 \cong X \cong G_2$. Define $\theta : G_1 \to G_2$ by $\theta = \theta_2\theta_1^{-1}$; then $X = \{g\theta(g) \mid g \in G_1\}$. (Note θ is only an isomorphism of abstract groups.)

Using this parametrisation of X, any (X, X)-double coset in G can be written as Xg_2X, with $g_2 \in G_2$. We now show that $Xg_2X = X\,\mathrm{Cl}_{G_2}(g_2)$.

Certainly, the right-hand side is contained in the left, so take $xg_2y \in Xg_2X$. Let $x = a\theta(a)$, $y = b\theta(b)$, for a, $b \in G_1$. Then, $xg_2y = ab\theta(ab)\theta(b)^{-1}g_2\theta(b)$, which is in $X\ \mathrm{Cl}_{G_2}(g_2)$ as claimed.

Now, suppose Xg_2X is dense in G, with $g_2 \in G_2$. Since $Xg_2X = X\ \mathrm{Cl}_{G_2}(g_2)$, the product morphism $\mu : X \times \overline{\mathrm{Cl}_{G_2}(g_2)} \to G$ is dominant. So, $\dim G \le \dim X + \dim \mathrm{Cl}_{G_2}(g_2) \le \dim X + \dim G_2 = \dim G$, and equality must hold throughout. It follows that $\mathrm{Cl}_{G_2}(g_2)$ is dense in G_2, a contradiction since G_2 has no dense conjugacy class. ∎

Now it is just a matter of assembling the pieces to prove Theorem 1. Consider the case where G is simple and X is an arbitrary reductive subgroup. If $\dim X < \frac{1}{2} \dim G$, then as $\dim XgX \le 2 \dim X < \dim G$, there is certainly no dense double coset in G. So, suppose $\dim X \ge \frac{1}{2} \dim G$. By Theorem 2(iv) and Lemma 0.1, there is no dense (X^0, X^0)-double coset in G. Hence, as X^0 has finite index in X, there is no dense (X, X)-double coset in G.

Next, let G be semisimple, and assume moreover that $Z(G) = 1$. So, $G = G_1G_2 \cdots G_n$, a direct product of simple factors. Use induction on n, the case $n = 1$ following by the preceding paragraph. Consider the projections $G \to G_{i_1} \cdots G_{i_k}$, where $1 \le i_1 < \cdots < i_k \le n$ and $k < n$. With $n > 2$, the argument of Lemma 3.1 shows that not all of these projections can be surjective, when induction gives the result. For $n = 2$, induction again applies unless both projections are surjective, when the result is just Lemma 3.1.

Finally, let G be an arbitrary connected reductive group. By Corollary 1.3(iii), we may assume $Z(G) \le X$. Thus, $X/Z(G)$ is a proper subgroup of the centreless semisimple group $G/Z(G)$, so has no dense double coset. The general result now follows by pulling back to show that X has no dense double coset in G.

The second statement of Theorem 1 follows immediately, completing the proof.

ACKNOWLEDGMENTS

We thank Martin W. Liebeck for suggesting the problem and for all his advice and support. This work was supported by an EPSRC grant.

REFERENCES

[ABS] H. Azad, M. Barry, and G. M. Seitz, On the structure of parabolic subgroups, *Comm. Algebra* **18** (1990), 551–562.

[Hab] W. Haboush, Reductive groups are geometrically reductive, *Ann. of Math.* **102** (1975), 67–84.

[Hum] J. E. Humphreys, "Linear Algebraic Groups," 3rd ed., Springer-Verlag, New York, 1987.

[I] N. F. J. Inglis, The embedding $O(2m, 2^k) \leq Sp(2m, 2^k)$, *Arch. Math.* **54** (1990), 327–330.

[Law] R. Lawther, Folding actions, *Bull. London Math. Soc.* **25** (1993), 132–144.

[LSS] M. W. Liebeck, J. Saxl, and G. M. Seitz, Factorizations of simple algebraic groups, to appear.

[Ri] R. W. Richardson, Affine coset spaces of reductive algebraic groups, *Bull. London Math. Soc.* **9** (1977), 38–41.

[Se] G. M. Seitz, Small rank permutation representations of finite Chevalley groups, *J. Algebra* **28** (1974), 508–517.